Contents

We want to give a modification of the Sweedler semantics [**?**] that will work for Hilbert spaces, and therefore allow for an involutive dual.

Sweedler Semantics

Consider the following map (functor) on $\mathcal{V}$ the category of (for the purposes of this document $\mathbb{R}eal$) vector spaces:

$$
\begin{aligned}
! : \mathcal{V} &\to \mathcal{V} \\
V &\to \bigoplus_{v \in V} Sym(V)
\end{aligned}
$$

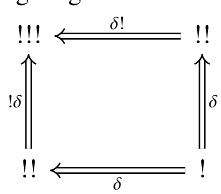
(And doing something on morhpisms) In [**?**] it is shown that this map, along with

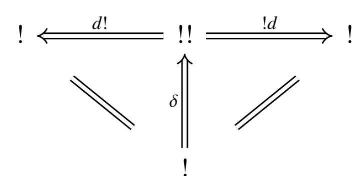
define sym

$$
\begin{aligned}
\delta_V \; &: !V \to !!V \\
|v_1, ..., v_s\rangle &\mapsto \sum_{C_1,...,C_\ell \in \mathcal{P}_{[s]}} ||v_{C_1}\rangle_P, ..., |v_{C_\ell}\rangle_P\rangle_{|\emptyset\rangle_P}
\end{aligned}
$$

$$
\begin{aligned}
d_V \; &: !V \to V \\
|v_1, ..., v_s\rangle_P &\mapsto \begin{cases} P & \text{s=0} \\ v_1 & \text{s=1} \\ 0 & \text{else} \end{cases}
\end{aligned}
$$

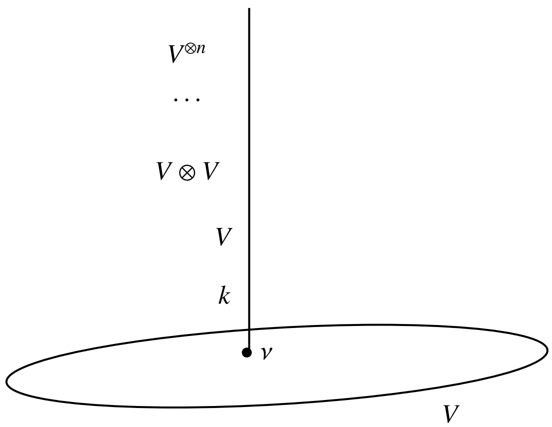
Form a comonad on $\mathcal{V}$, that is to say there are natural transformations d, δ, which are component wise d_V and δ_V above, that make the following diagrams commute:





There are also two further functions making this into a coalgebra modality.

add



Restriction to Inner Product Spaces

0.2.1 Bang Functor

All innerproduct spaces are vector spaces so it makes sense to act on one via !

To define an inner product on the symmetric algebra first note that an arbitrary element of $!V$ is of the form

$$([(v_{0,i,v} \otimes \cdots \otimes v_{i,i,v})_{i \in \mathbb{N}_0}]_v)_{v \in V}$$

where $v_{a,b,c} \in V$ for every appropriate $a \leq b \in \mathbb{N}_0, c \in V$ Moreover only finitely many of the equivilence classes are non-zero (the zero element being the equivilence class of the identically zero sequence) and only finitely many of the tensors $v_{0,i,v} \otimes \cdots \otimes v_{i,i,v}$ are non-zero for any v.

The natural inner product on the direct sum of the symmetric algebras (extending those used for the direct sum and tensor) is then

$$\langle ([(v_{0,i,v} \otimes \cdots \otimes v_{i,i,v})_{i \in \mathbb{N}_0}]_v)_{v \in V}, ([(u_{0,i,v} \otimes \cdots \otimes u_{i,i,v})_{i \in \mathbb{N}_0}]_v)_{v \in V} \rangle_! = \sum_{v \in V} \sum_{i \in \mathbb{N}_0} \sum_{\sigma_1, \sigma_2 \in \mathcal{S}_i} \prod_{j=0}^{i} \langle v_{\sigma_1(j),i,v}, u_{\sigma_2(j),i,v} \rangle$$

Where $\mathcal{S}_i$ is the symetric group on the set of i elements and an element acts on a tensor via

$$\sigma(v_1 \otimes \ldots \otimes v_n) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}$$

Note that because only finitely many of the summands may be nonzero this is a finite sum and the properties of an inner product follow from the inductive definition. .

check them I guess

So infact ! restricts to a functor on innerproduct spaces

$$\begin{aligned} !_i : \underline{Inner}_k &\to \underline{Inner}_k \\ (V, \langle, \rangle) &\to (!V, \langle, \rangle_!) \end{aligned}$$

I still dont know what it does to morphisms

0.2.2 Comonad

To have any reasonable hope of calling what we are doing a modification or restriction of the Sweedler semantics we would hope that the d and δ maps above restrict to morphisms of innerproduct spaces. Both maps are linear (as vector space morphisms) so it remains for them to be continuous. By a well known theorem (in characteristic 0) for normed spaces, linear and continuous is equvilent to linear and bounded, where bounded for a linear map $L : X \to Y$ means

$$\exists M > 0 \ \ \forall x \in X \quad ||L(x)|| \leq M||x||$$

Lemma. *d_V is not bounded.*

Proof. If there was such an $M > 0$ then for all $p \in V$

$$||d_V|\emptyset\rangle_p|| = ||p|| \leq M||\emptyset||$$

Which is a contradiction.

In particular any innerproduct has the following $\langle x, x\rangle = 0 \iff x = 0$ and hence by linearity must be unbounded $||\alpha x|| = \alpha||x||$ because (as long as there is more than a single element) some x will have a positive norm.

Lemma. *δ_V is not bounded.*

Proof. We produce an example of a sequence of basis vecotors such that the norm of their image is increasing to infinity but M times their norm is a constant. By the argument at the end of the other proof if the normed space is nondegenerate then there must be an element of every real valued norm, so in particular for a given V there is an element with norm one. Then let this element be x, we have by definition of the norm on the tensor of two innerproduct spcaes (i.p of tensors is product of i.p) that $||x^{\otimes k}|| = ||x||^k = 1$. Hence $M||x^{\otimes k}|| = M$. On the other hand

$$\begin{aligned}||\delta_V|x^{\otimes s}\rangle_P|| &= || \sum_{\{C_1,\dots,C_\ell\}\in\mathcal{P}_{[S]}} ||x_{C_1}\rangle_P, \dots, |x_{C_\ell}\rangle_P\rangle_{|\emptyset\rangle_P}|| \\ &\geq || \sum_{\{C_1,\dots,C_\ell\}\in\mathcal{P}'_{[S]}} ||x_{C_1}\rangle_P, \dots, |x_{C_\ell}\rangle_P\rangle_{|\emptyset\rangle_P}||\end{aligned}$$

Where $\mathcal{P}'$ are the partitions of $[s]$ that are ordered, i.e. if $a \in C_i$ then $\exists b \in C_i$ such thtat $a = b \pm 1$.

$$\begin{aligned}&= \langle \sum_{\{C_1,\dots,C_\ell\}\in\mathcal{P}'_{[S]}} ||x_{C_1}\rangle_P, \dots, |x_{C_\ell}\rangle_P\rangle_{|\emptyset\rangle_P}, \sum_{\{C_1,\dots,C_\ell\}\in\mathcal{P}'_{[S]}} ||x_{C_1}\rangle_P, \dots, |x_{C_\ell}\rangle_P\rangle_{|\emptyset\rangle_P}\rangle \\ &= \sum_{\{C_1,\dots,C_\ell\}\in\mathcal{P}'_{[S]}} \sum_{\{C'_1,\dots,C'_\ell\}\in\mathcal{P}'_{[S]}} \langle ||x_{C_1}\rangle_P, \dots, |x_{C_\ell}\rangle_P\rangle_{|\emptyset\rangle_P}, ||x_{C'_1}\rangle_P, \dots, |x_{C'_\ell}\rangle_P\rangle_{|\emptyset\rangle_P}\rangle \\ &= \sum_{\{C_1,\dots,C_\ell\}\in\mathcal{P}'_{[S]}} \sum_{\{C'_1,\dots,C'_\ell\}\in\mathcal{P}'_{[S]}} \mathbb{I}(\ell = \ell')\mathbb{I}(|C_i| = |C'_i|\ \forall i \in [\ell])\langle ||x_{C_1}\rangle_P, \dots, |x_{C_\ell}\rangle_P\rangle_{|\emptyset\rangle_P}, ||x_{C'_1}\rangle_P, \dots, |x_{C'_\ell}\rangle_P\rangle_{|\emptyset\rangle_P}\rangle \\ &= \sum_{\{C_1,\dots,C_\ell\}\in\mathcal{P}'_{[S]}} \langle ||x_{C_1}\rangle_P, \dots, |x_{C_\ell}\rangle_P\rangle_{|\emptyset\rangle_P}, ||x_{C_1}\rangle_P, \dots, |x_{C_\ell}\rangle_P\rangle_{|\emptyset\rangle_P}\rangle\end{aligned}$$

Because if $\ell = \ell'$ there are the same number of boxes and if they each have the same size and our indecies are increasing, then the two partitions are equal.

$$\begin{aligned}&= \sum_{\{C_1,\dots,C_\ell\}\in\mathcal{P}'_{[S]}} |||x_{C_1}\rangle_P, \dots, |x_{C_\ell}\rangle_P\rangle_{|\emptyset\rangle_P}|| \\ &= \sum_{\{C_1,\dots,C_\ell\}\in\mathcal{P}'_{[S]}} \prod_{i=1}^{\ell} |||x_{C_i}\rangle_P|| \\ &= \sum_{\{C_1,\dots,C_\ell\}\in\mathcal{P}'_{[S]}} \prod_{i=1}^{\ell} |||x^{\otimes|C_i|}\rangle_P|| \\ &= \sum_{\{C_1,\dots,C_\ell\}\in\mathcal{P}'_{[S]}} \prod_{i=1}^{\ell} ||x||^{|C_i|} \\ &= \sum_{\{C_1,\dots,C_\ell\}\in\mathcal{P}'_{[S]}} 1 \\ &= |\{\text{ ordered partitions of } [s]\}|\end{aligned}$$

Since the cardinality of the collection of ordered partitions of $[s]$ is unbounded (as a function in s) we can conclude that $||\delta_V|x^{\otimes s}\rangle_P|| \geq |\{\text{ ordered partitions of } [s]\}|$ is also unbounded as a linear function.

I think that this proof is essentially correct but I corrected the definition of the innerproduct and I think that there was some error in here

Patching the Leaks

0.3.1 Identifying the Problems

Ok so $!_i$ doesnt work because the comonad functions are not continuous however there are several degrees of freedom in this definitino that we can change in order to potentially get an innerproduct space. First we can alter the object:

$$\bigoplus_{v\in V, \|v\|\leq p_1} Sym^{\leq p_2}(V)_{\leq p_3}$$

- p_1: We restrict the size of the vectors over which we place fibre.
- p_2: Restrict the height of the tensor algebra
- p_3: Restrict the size of each of the elements that we tensor

And scale the inner product itself:

$$\langle([(v_{0,i,v}\otimes\cdots\otimes v_{i,i,v})_{i\in\mathbb{N}_0}]_v)_{v\in V}, ([(u_{0,i,v}\otimes\cdots\otimes u_{i,i,v})_{i\in\mathbb{N}_0}]_v)_{v\in V}\rangle_! = \sum_{v\in V}\rho_v\sum_{i\in\mathbb{N}_0}\sum_{\sigma_1,\sigma_2\in\mathcal{S}_\rangle}\prod_{j=0}^{i}\langle v_{\sigma_1(j),i,v}, u_{\sigma_2(j),i,v}\rangle$$

For some (positive?) constants ρ_v.

We can see from the proofs of unboundedness that in the case of d the failure came because the inner product was indifferent to the point at which you took the vacuum and the norms of the points p were unbounded. This can be solved in two ways

- Restrict the norm of the points over which we place a fibre, p_1
- Scale the norm that we use for each fibre, ρ_v

Im told that the latter is unmotivated from the perspective of this as a jet bundle so we will investigate the former more thoroughly first.

Now for δ the immediate problem was that:

- We could always find tensors of size one in each fibre and at each level
- The "height" of the tensor was unbounded

In this case it seems like the more fundamental issue was that the height of the tensor was unbounded, and it seems sufficient to control this to stop the issue.

0.3.2 Some Calculations

The norm of an arbitrary element can be simplified to

$$\begin{aligned}\|([(v_{0,i,v}\otimes\cdots\otimes v_{i,i,v})_{i\in\mathbb{N}_0}]_v)_{v\in V}\| &= \langle([(v_{0,i,v}\otimes\cdots\otimes v_{i,i,v})_{i\in\mathbb{N}_0}]_v)_{v\in V}, ([(v_{0,i,v}\otimes\cdots\otimes v_{i,i,v})_{i\in\mathbb{N}_0}]_v)_{v\in V}\rangle \\ &= \sum_{v\in V}\sum_{i\in\mathbb{N}_0}\sum_{\sigma_1,\sigma_2\in\mathcal{S}_i}\prod_{j=0}^{i}\langle v_{\sigma_1(j),i,v}, v_{\sigma_2(j),i,v}\rangle\end{aligned}$$

Note that if the $v_{a,b,c}$ are all orthogonal then this norm is immediately zero. Also notice that the inner product is always in the same fibre.

Note a subtlety that the notation of the input is deceptive because for both indexes 0 and 1 there are no tensors, 0 being the field element and 1 being the element of the vector space (or we associate $k\otimes_k V\cong V$). Hence in the case of the entries being zero for the indexes $i\geq 2$ we have the norm being

$$\|(\alpha_v, v_v)_{v\in V}\| = \sum_{v\in V}\left(|\alpha_v|^2 + \|v_v\|^2\right)$$

Where we have safely ignored the action of the tensor and equivalence class because they are not present. Also notice that this is zero iff the input is zero (by the norm axioms).

d Function We are given the functions d and δ in terms of their action on a basis, unfortunately this is not sufficient to prove boundedness so we need to understand how they act on a general vector. This computation is to that end

$$\begin{aligned} d_V([(v_{0,i,v} \otimes \cdots \otimes v_{i,i,v})_{i\in\mathbb{N}_0}]_v)_{v\in V} &= \sum_{v\in V} d_V(v_{0,i,v} \otimes \cdots \otimes v_{i,i,v})_{i\in\mathbb{N}_0} \\ &= \sum_{v\in V}\sum_{i\in\mathbb{N}_0} d_V(v_{0,i,v} \otimes \cdots \otimes v_{i,i,v}) \\ &= \sum_{v\in V}\sum_{i\in\mathbb{N}_0} d_V|v_{0,i,v} \otimes \cdots \otimes v_{i,i,v}\rangle_v \\ &= \sum_{v\in V}\sum_{i\in\mathbb{N}_0} \delta_{i=0}\delta_{v_{0,0,v}\neq 0} v + \delta_{i=1} v_{0,1,v}\cdot v_{1,1,v} \\ &= \sum_{v\in V} \delta_{v_{0,0,v}\neq 0} v + v_{0,1,v}\cdot v_{1,1,v} \\ &= \sum_{v\in V} \delta_{v_{0,0,v}\neq 0} v + v_{1,1,v} \end{aligned}$$

Where we identify $\alpha \otimes v = 1 \otimes \alpha v$ and because this is arbitrary and the $v_{0,1,v}$ is a field element we can ignore it (absorb it into the general 1,1 entry).

And then the norm of this result is

$$\begin{aligned} \|d_V([(v_{0,i,v} \otimes \cdots \otimes v_{i,i,v})_{i\in\mathbb{N}_0}]_v)_{v\in V}\| &= \sum_{v\in V}\sum_{u\in V} \langle \delta_{v_{0,0,v}\neq 0} v + v_{0,1,v}\cdot v_{1,1,v}, \delta_{v_{0,0,u}\neq 0} u + v_{0,1,u}\cdot v_{1,1,u}\rangle \\ &= \sum_{v\in V}\sum_{u\in V} \delta_{v_{0,0,v}\neq 0}\delta_{v_{0,0,u}\neq 0}\langle v, u\rangle + \delta_{v_{0,0,v}\neq 0} v_{0,1,u}\cdot\langle v, v_{1,1,u}\rangle + \delta_{v_{0,0,u}\neq 0} v_{0,1,v}\cdot\langle v_{1,1,v}, u\rangle \\ &\qquad + v_{0,1,v}v_{0,1,u}\cdot\langle v_{1,1,v}, v_{1,1,u}\rangle \\ &= \sum_{v\in V}\sum_{u\in V} \delta_{v_{0,0,v}\neq 0}\delta_{v_{0,0,u}\neq 0}\langle v, u\rangle + \delta_{v_{0,0,v}\neq 0}\langle v, v_{1,1,u}\rangle + \delta_{v_{0,0,u}\neq 0}\langle v_{1,1,v}, u\rangle + \langle v_{1,1,v}, v_{1,1,u}\rangle \end{aligned}$$

Hence

$$\|d_V(\alpha_v, v_v)_{v\in V}\| = \sum_{u,v\in V} \delta_{\alpha_v,\alpha_u\neq 0}\langle v, u\rangle + \delta_{\alpha_v\neq 0}\langle v, v_u\rangle + \delta_{\alpha_u\neq 0}\langle v_v, u\rangle + \langle v_v, v_u\rangle$$

So if all the scalars are zero we get

$$\|d_V(0, v_v)_{v\in V}\| = \sum_{u,v\in V} \langle v_v, v_u\rangle$$

And if all the vectors are zero we get

$$\|d_V(\alpha_v, 0)_{v\in V}\| = \sum_{u,v\in V} \delta_{\alpha_v,\alpha_u\neq 0}\langle v, u\rangle$$

I believe one can see already that none of the restrictions will allow this to work. I will try and give an example below.

delta Function Now calculuate for δ on arbitrary entries:

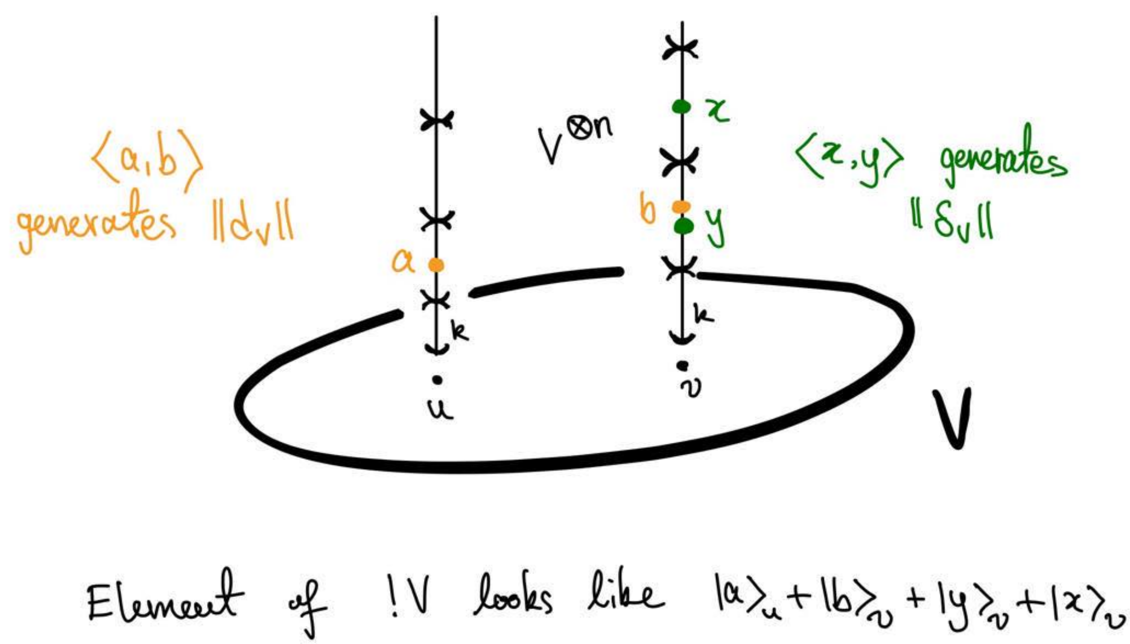
$$\delta([(v_{0,i,v} \otimes \cdots \otimes v_{i,i,v})_{i\in\mathbb{N}_0}]_v)_{v\in V} = \sum_{v\in V}\sum_{i\in\mathbb{N}_0}\sum_{C_1,\ldots,C_\ell\in\mathcal{P}_{[i]}} ||v_{C_1,i,v}\rangle_v, \ldots, |v_{C_\ell,i,v}\rangle_v\rangle_{|\emptyset\rangle_v}$$

Then taking the norm we get

$$\begin{aligned}
\|\delta([(v_{0,i,v} \otimes \cdots \otimes v_{i,i,v})_{i\in\mathbb{N}_0}]_v)_{v\in V}\| &= \sum_{v,u\in V} \sum_{i,j\in\mathbb{N}_0} \sum_{C_1,\dots,C_\ell\in\mathcal{P}_{[i]}} \sum_{D_1,\dots,D'_\ell\in\mathcal{P}_{[j]}} \langle \| |v_{C_1,i,v}\rangle_v, \dots, |v_{C_\ell,i,v}\rangle_v\rangle_{|\emptyset\rangle_v}, \| |v_{D_1,j,u}\rangle_u, \dots, |v_{D'_\ell,j,u}\rangle_u\rangle_{|\emptyset\rangle_u}\rangle \delta_{\ell=\ell'}\delta_{u=v} \\
&= \sum_{v\in V} \sum_{i,j\in\mathbb{N}_0} \sum_{C_1,\dots,C_\ell\in\mathcal{P}_{[i]}} \sum_{D_1,\dots,D_\ell\in\mathcal{P}_{[j]}} \sum_{\sigma_1,\sigma_2\in\mathcal{S}_\ell} \prod_{k=0}^{\ell} \langle |v_{C_{\sigma_1(k)},i,v}\rangle_v, |v_{D_{\sigma_2(k)},j,u}\rangle_v\rangle \delta_{|C_{\sigma_1(k)}|=|D_{\sigma_2(k)}|} \\
&= \sum_{v\in V} \sum_{i,j\in\mathbb{N}_0} \sum_{C_1,\dots,C_\ell\in\mathcal{P}_{[i]}} \sum_{D_1,\dots,D_\ell\in\mathcal{P}_{[j]}} \sum_{\sigma_1,\sigma_2\in\mathcal{S}_\ell} \prod_{k=0}^{\ell} \sum_{s_1,s_2\in\mathcal{S}_{|C_{\sigma_1(k)}|}} \prod_{m=0}^{|C_{\sigma_1(k)}|} \langle v_{s_1(\pi_m(C_{\sigma_1(k)})),i,v}, v_{s_2(\pi_m(D_{\sigma_2(k)})),j,u}\rangle \delta_{|C_{\sigma_1(k)}|=|D_{\sigma_2(k)}|} \\
&= \sum_{v\in V} \sum_{i\in\mathbb{N}_0} \sum_{\{C_1,\dots,C_\ell\},\{D_1,\dots,D_\ell\}\in\mathcal{P}_{[i]}} \sum_{\sigma_1,\sigma_2\in\mathcal{S}_\ell} \prod_{k=0}^{\ell} \sum_{s_1,s_2\in\mathcal{S}_{|C_{\sigma_1(k)}|}} \prod_{m=0}^{|C_{\sigma_1(k)}|} \langle v_{s_1(\pi_m(C_{\sigma_1(k)})),i,v}, v_{s_2(\pi_m(D_{\sigma_2(k)})),i,u}\rangle \delta_{|C_{\sigma_1(k)}|=|D_{\sigma_2(k)}|}
\end{aligned}$$

Where $\pi_m(X)$, $X \subseteq \mathbb{N}$ is the m'th entry when the values of X are ordered according to the standard order on $\mathbb{N}$. Note that we simplify from summing over all i and j to just i because of the condition that $\forall k \;\; |C_{\sigma_1(k)}| = |D_{\sigma_2(k)}|$, because if i and j are different it is not possible for this to be the case for all k, hence those terms vanish.

Again one can see that although we only compare elements of the same fibre, now we are mixing the "level" of the tensor that we compare. It is then easy to take an element that has zero norm, say orthogonal entries on each tensor, that is has non-zero norm under δ because the elements of the tensors between levels are not orthogonal. This again is only possible to block by restricting the height of the tensor to one.



Concrete Examples Recall that

$$\|(\alpha_v, v_v)_{v\in V}\| = \sum_{v\in V} \left(|\alpha_v|^2 + \|v_v\|^2\right)$$

and

$$\|d_V(0, v_v)_{v\in V}\| = \sum_{u,v\in V} \langle v_v, v_u\rangle$$

$$\|d_V(\alpha_v, 0)_{v\in V}\| = \sum_{u,v\in V} \delta_{\alpha_v,\alpha_u\neq 0}\langle v, u\rangle$$

So clearly the parameter p_2 will play no role, unless we restricted the height to ≤ 1 because the function only sees the first two levels as it were. Now assume that we set both $p_1, p_3 \leq 1$ (because it is a special case where inner producting elements will also have a bounded norm) i.e. Using Cauchy Schwartz

$$||a||, ||b|| \leq 1 \implies |\langle a, b\rangle|^2 \leq ||a||^2||b||^2 \leq 1$$

Assume that d_V is bounded then there is an $M \in \mathbb{N}$ such that for every input

$$||d_V(0, v_v)_{v\in V}|| = \sum_{u,v\in V}\langle v_v, v_u\rangle \leq M\sum_{v\in V}\langle v_v, v_v\rangle$$

$$\sum_{v\in V}\langle v_v, v_v\rangle + \sum_{u,u'\in V,u\neq u'}\langle v_u, v_{u'}\rangle \leq M\sum_{v\in V}\langle v_v, v_v\rangle$$

$$\sum_{u,u'\in V,u\neq u'}\langle v_u, v_{u'}\rangle \leq (M-1)\sum_{v\in V}\langle v_v, v_v\rangle$$

So if every v_v was the same, say v, regardless of the restriction we would have that

$$\sum_{u,u'\in V,u\neq u'}\langle v, v\rangle \leq (M-1)\sum_{v\in V}\langle v, v\rangle$$

$$|\{(u, u') \in V^2 : v_u \neq 0, v_{u'} \neq 0, u \neq u'\}|\langle v, v\rangle \leq (M-1)\sum_{v\in V}\langle v, v\rangle$$

Which is a contradiction because we can always increase the number of nonzero entries such that $|\{(u, u') \in V^2 : v_u \neq 0, v_{u'} \neq 0, u \neq u'\}| > M$. (A similar and simpler contradiction can be arrived at with all the vectors being zero, that is similarly uneffected by the restrictions).

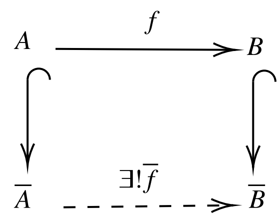
Lifting to Hilbert Spaces

This is a shame because we would have been done if d_V and δ_V restricted to innerproduct morphisms. We will explain why here:

If these maps were morphisms then the functoriality of $!_{fin}$ would be sufficient for them being a comonad on *Inner*

Lemma. *Completion is a functor from Inner to $\mathcal{H}$ the category of Hilbert spaces.*

Proof. The closure of a space has the following universal property in *Inner*



From this it follows immediately that $\overline{f \circ g} = \bar{f} \circ \bar{g}$ and that $i\bar{d}_V = id_{\bar{V}}$ by simply observing that these maps fit into the diagram.

And now the functoriality of $\overline{(-)}$ is sufficient for $(\overline{(-)} \circ !_{fin}, \overline{d}, \overline{\delta})$ to be a comonad on $\mathcal{H}$, because functors take commuting diagrams to commuting diagrams.